MATHEMATICS

ON THE BEHAVIOUR OF THE KALMAN-BUCY ESTIMATE IF THE INVOLVED WIENER-LÉVY PROCESSES ARE APPROXIMATED BY PHYSICALLY REALIZABLE PROCESSES

BY

P. A. RUYMGAART

(Communicated by Prof. R. TIMMAN at the meeting of January 29, 1972)

ABSTRACT

It is noteworthy that the possibility of computing Kalman-Bucy estimates depends entirely on the mathematical properties of the Wiener-Lévy processes figuring in the observations, whereas no observation device ever could generate processes of that kind. However, there are physically realizable processes arbitrarily close to Wiener-Lévy processes. In this paper the consequences are investigated if the Wiener-Lévy processes in the Kalman-Bucy filter are replaced by realizable approximations. The effect is not simply that of perturbing some values in certain formulae, since the whole computation scheme of Kalman and Bucy breaks down. It is shown that estimates of Kalman-Bucy type, based on a finite number of observations, are stable with respect to the above sketched operations. A more or less controversial result is obtained if the number of observations is infinite.

1. PREREQUISITES

All random variables in this paper are real-valued and belong to a real centered Gaussian Hilbert space H, of which $\{\Omega, \mathscr{A}, P\}$ is the underlying probability space. If $\xi, \eta, \ldots$ belong to H, then $E\xi=0$, $E\xi^2<\infty$, $E\xi\eta=(\xi, \eta)$ is the inner product in H and $\|\xi\|=\sqrt{E\xi^2}$ the norm. As equivalent random variables are identified, the addition "a.s." is omitted in identities in H. $\xi \perp \eta$ means $E\xi\eta=0$, or equivalently, ξ and η are stochastically independent since H is centered Gaussian. All limit operations in H are meant in quadratic mean (in q.m.), i.e. in the strong sense. So $\xi \to \eta$ in q.m. means $\|\xi-\eta\| \to 0$. $[0, T]$ is an interval of the real line, and all stochastic processes in this paper are understood to be mappings of $[0, T]$ into H. Derivatives and integrals of stochastic processes are limits in q.m. of Cauchy sequences in H. In the exceptional case that the trajectories of a process are examined, this process may and will be assumed to be separable in Doob's sense. "Vector" stands for "column-vector". Identity matrices are denoted by I_N, N being the number of rows and columns. The superscript "$^{\mathrm{T}}$" means "transpose of". We recall and state

DEFINITION 1: $\beta(t)$, $t \in [0, T]$, is an M-dimensional Wiener-Lévy process whose components $\beta_j(t)$, $j=1, \ldots, M$, are mappings of $[0, T]$ into H.

Hence $E\beta(t)=0$. The covariance matrix satisfies $E\beta(s)\beta^{\mathsf{T}}(t)=\int_0^m B(u)du$, $m=\min(s,t)$, $(s,t)\in[0,T]^2$. The entries of the $M\times M$-matrix $B(u)$ are continuous mappings of $[0,T]$ into the real line. At each $u\in[0,T]$, $B(u)$ is necessarily symmetric and definite non-negative. However, it is moreover assumed that

$$(1)\quad \begin{cases} B(u)>0 \text{ at each } u\in[0,T]. \text{ Then there is a number } e>0 \text{ such} \\ \text{that } B(u)\geqq eI_M \text{ at each } u\in[0,T]. \end{cases}$$

We recall the following consequences: $\beta(0)=0$, the components $\beta_j(t)$ are continuous in q.m. on $[0,T]$ and their increments $\beta_j(t)-\beta_j(s)$ on disjunct intervals $[s_1,t_1)$ and $[s_2,t_2)$ are orthogonal. However, $\beta_j(t)$ is not differentiable in q.m. Almost all trajectories of $\beta_j(t)$ are continuous on $[0,T]$, but not differentiable and not of bounded variation on any sub-interval of $[0,T]$. Hence $\beta(t)$ is not "physically realizable".

The following statement shows that $\beta(t)$ may be closely approximated by "physically realizable" processes. Details and proofs may be found in [1].

THEOREM 1: To $\beta(t)$ in definition 1 there is a sequence

$$(2)\qquad \{\beta(m,t),\ t\in[0,T],\ m=1,2,\ldots\}$$

with the following properties:

i) The components $\beta_j(m,t)$, $j=1,\ldots,M$, of $\beta(m,t)$ are mappings of $[0,T]$ into H, entailing $E\beta_j(m,t)=0$.

$\beta_j(m,t)$ is defined on $\beta_j(s)$, $s\in[0,T]$. And $\beta_j(m,0)=0$. For instance, $\beta_j(m,t)$ may be constructed by means of convolutions of $\beta_j(s)$ with certain real valued functions. It may also be a polygon with vertices on the curve $\beta_j(s)$, $s\in[0,T]$, and hence at all $t\in[0,T]$, $\beta_j(m,t)$ may be defined on one and the same finite set of random variables of the process $\beta_j(s)$, $s\in[0,T]$.

ii) Almost all trajectories of $\beta_j(m,t)$ are arbitrarily often differentiable on $[0,T]$.

$\beta_j(m,t)$ is arbitrarily often differentiable in q.m. on $[0,T]$.

(In case $\beta_j(m,t)$ is a polygon, the derivatives at the vertices should be equal to 0).

iii) As $m\to\infty$, $\beta_j(m,t)\to\beta_j(t)$ in q.m., uniformly in $t\in[0,T]$, entailing $E\beta_j(m,s)\beta_k(m,t)\to E\beta_j(s)\beta_k(t)$ uniformly in $(s,t)\in[0,T]^2$.

And, as $m\to\infty$, $\beta_j(m,t)\to\beta_j(t)$ a.s. on $[0,T]$ in the sense that for any $\varepsilon>0$,

$$P[\bigcup_{m'\geqq m}[\sup_{t\in[0,T]}|\beta_j(m',t)-\beta_j(t)|\geqq\varepsilon]]\downarrow 0.$$

iv) The entries of the covariance matrices $E\beta(m,s)\beta^{\mathsf{T}}(m,t)$ are of bounded variation as functions of (s,t) on $[0,T]^2$, uniform in m.

v) If $0 \leqq u \leqq v \leqq w \leqq T$, it is because of the continuity of the inner product in H, and owing to iii), that as $m \to \infty$,

$$E\{\beta_j(m, v)-\beta_j(m, u)\}\{\beta_k(m, w)-\beta_k(m, v)\} \to$$

$$E\{\beta_j(v)-\beta_j(u)\}\{\beta_k(w)-\beta_k(v)\}=0, \; j, k=1, \ldots, M.$$

The increments of $\beta_j(m, t)$ on disjunct closed intervals of $[0, T]$ are orthogonal if m is sufficiently large.

Let $\alpha(t)$, $t \in [0, T]$, with components $\alpha_i(t)$ in H, $i=1, \ldots, N$, be an N-dimensional Wiener-Lévy process with properties analogous to those of $\beta(t)$ in definition 1. Concerning $\alpha(t)$, a condition similar to (1) does not need to be satisfied. It is assumed that $\alpha(t)$ and $\beta(t)$ are stochastically independent, $E\alpha(u)\beta^{\mathrm{T}}(v)=0$, $(u, v) \in [0, T]^2$.

Let γ be an N-vector with components γ_i in H, $i=1, \ldots, N$, and such that $E\alpha(s)\gamma^{\mathrm{T}}=0$, $E\beta(s)\gamma^{\mathrm{T}}=0$, $s \in [0, T]$.

Let $A(s)$, $s \in [0, T]$, be an $N \times N$-matrix whose entries are continuous real functions of s.

Let us consider the N-dimensional system of integral equations in q.m.

$$\xi(t)=\gamma+\int_0^t A(s)\xi(s)ds+\alpha(t), \; t \in [0, T]. \tag{3}$$

It is well known that this system has a unique solution $\xi(t)$, whose components $\xi_i(t)$ are mappings of $[0, T]$ into H.

Let S_t be a subset of $[0, T]$, varying with t. Given $t \in [0, T]$, $\xi(t)$ is "observed" as the M-vector

$$\zeta(s)=\eta(s)+\beta(s) \text{ at each } s \in S_t,$$

where

$$\eta(s)=\int_0^s F(u)\xi(u)du,$$

$F(u)$ being an $M \times N$-matrix whose entries are continuous real functions of $u \in [0, T]$. It is seen that the components $\eta_j(s)$ and $\zeta_j(s)$, $j=1, \ldots, M$, of $\eta(s)$ and $\zeta(s)$ belong to H, and it follows that

$$E\eta(u)\beta^{\mathrm{T}}(v)=0, \; (u, v) \in [0, T]^2. \tag{4}$$

We are interested in the conditional expectation $\hat{\xi}(t|S_t)$ of $\xi(t)$, given the observations $\zeta(s)$, $s \in S_t$. Since all random variables involved belong to the centered Gaussian system H, the components $\hat{\xi}_i(t|S_t)$ of $\hat{\xi}(t|S_t)$ are identical to the linear minimum variance estimate of the corresponding components $\xi_i(t)$, given $\zeta(s)$ on S_t, i.e. $\hat{\xi}_i(t|S_t)$ is the orthogonal projection of $\xi_i(t)$, $i=1, \ldots, N$, onto the closed linear subspace

$$H[C(S_t)]$$

of H, generated by the elements of the class

$$C(S_t)=\{\zeta_j(s), j=1, \ldots, M, \; s \in S_t\}.$$

Depending on the position of t and S_t in $[0, T]$, $\hat{\xi}(t|S_t)$ is an interpolated, filtered or extrapolated random N-vector.

If $S_t=[0, t]$, $\hat{\xi}(t|S_t)$ is the Kalman-Bucy estimate. It may be computed as the solution of a system of stochastic differential equations, thanks to the circumstance that it may be represented as an integral, whereas this representation is available owing to the presence of the Wiener-Lévy process $\beta(t)$ in the observations, see [1]. A comparable situation arises in the estimation theory of stationary processes, owing to the spectral representation of these processes.

Let the sequence

$$\{\alpha(n, t),\ t \in [0, T],\ n=1, 2, \ldots\}$$

be related to $\alpha(t)$ as the sequence (2) is related to $\beta(t)$. We shall now describe the above estimation procedure in case $\alpha(t)$ and $\beta(t)$ are replaced by the physically realizable approximations $\alpha(n, t)$ and $\beta(m, t)$. Thus, instead of (3) we now consider the system of differential equations in q.m.

$$\frac{d}{dt}\xi(n,t)=A(t)\xi(n,t)+\frac{d}{dt}\alpha(n, t)$$

with initial condition

$$\xi(n, 0)=\gamma.$$

To this system there is a unique solution $\xi(n, t)$ whose components $\xi_i(n, t)$, $i=1, \ldots, N$, are mappings of $[0, T]$ into H.

Given $t \in [0, T]$, $\xi(n, t)$ is "observed" as the M-vector

$$\zeta(n, m, s)=\eta(n, s)+\beta(m, s) \text{ at each } s \in S_t,$$

where

$$\eta(n, s)= \int_0^s F(u)\xi(n, u)du. \tag{5}$$

The components $\eta_j(n, s)$ and $\zeta_j(n, m, s)$, $j=1, \ldots, M$, of $\eta(n, s)$ and $\zeta(n, m, s)$ are seen to belong to H. By virtue of (5) and property i) in theorem 1,

$$\zeta(n, m, 0)=0 \tag{6}$$

and, also owing to (4),

$$E\eta(n, u)\beta^{\mathrm{T}}(m, v)=0,\ (u, v) \in [0, T]^2. \tag{7}$$

We are concerned with the estimate $\hat{\xi}(n, m, t|S_t)$ whose components $\xi_i(n, m, t|S_t)$, $i=1, \ldots, N$, are the conditional expectation of the corresponding $\xi_i(n, t)$, given the class

$$C(n, m, S_t)=\{\zeta_j(n, m, s),\ j=1, \ldots, M,\ s \in S_t\}.$$

If $H[C(n, m, S_t)]$ is the closed linear subspace of H, generated by the elements of $C(n, m, S_t)$, then analogously to the non-perturbed case,

$\hat{\xi}_i(n, m, t|S_t)$ is the orthogonal projection of $\xi_i(n, t)$ onto $H[C(n, m, S_t)]$.

If $S_t=[0, t]$, or if S_t is any infinite subset of $[0, T]$, there is in general no suitable representation of $\hat{\xi}(n, m, t|S_t)$ and there are no reliable methods for computing this estimate.

2. Preliminary remarks on $\hat{\xi}(n, m, t|S_t)$ as a function of n and m

The question arises how close is $\hat{\xi}(n, m, t|S_t)$ to $\hat{\xi}(t|S_t)$. For lack of equations containing $\hat{\xi}(n, m, t|S_t)$ explicitely if S_t is an infinite set, formulae of Kalman-Bucy type are in general not available in this case.

Lemma 1: As $n \to \infty$, then $\xi_i(n, t) \to \xi_i(t)$ in q.m., uniformly in $t \in [0, T]$, $i=1, \ldots, N$.

The easy proof may be found in [1].

Lemma 2: $\zeta_j(n, m, s) \to \zeta_j(s)$ in q.m. as $n, m \to \infty$, uniformly in $s \in [0, T]$, $j=1, \ldots, M$.

Proof: On account of (5) and lemma 1, $\eta_j(n, s) \to \eta_j(s)$ in q.m. uniformly in $s \in [0, T]$. The assertion follows with the aid of property iii) in theorem 1 since $\zeta_j(n, m, s)=\eta_j(n, s)+\beta_j(m, s)$.

Let t and S_t be fixed and let

$$\mathscr{P} \text{ and } \mathscr{P}(n, m)$$

be the orthogonal projectors of H onto

$$H[C(S_t)] \text{ and } H[C(n, m, S_t)]$$

respectively. Then

$$\hat{\xi}_i(t|S_t)=\mathscr{P}\xi_i(t) \text{ and } \hat{\xi}_i(n, m, t|S_t)=\mathscr{P}(n, m)\xi_i(n, t).$$

Definition 2: $\hat{\xi}(t|S_t)$ is stable if

$$\hat{\xi}_i(n, m, t|S_t) \to \hat{\xi}_i(t|S_t) \text{ in q.m. as } n, m \to \infty,$$

i.e. if

$$\mathscr{P}(n, m)\xi_i(n, t) \to \mathscr{P}\xi_i(t) \text{ in q.m. as } n, m \to \infty, \; i=1, \ldots, N.$$

Lemma 3: In order that $\hat{\xi}(t|S_t)$ be stable, it is sufficient that

$$\mathscr{P}(n, m)\varphi \to \mathscr{P}\varphi \text{ in q.m. as } n, m \to \infty, \text{ for all } \varphi \in H.$$

Proof:

$$\|\mathscr{P}(n, m)\xi_i(n, t)-\mathscr{P}\xi_i(t)\| \leqq \|\mathscr{P}(n, m)\| \cdot \|\xi_i(n, t)-\xi_i(t)\|+\|\mathscr{P}(n, m)\xi_i(t)-\mathscr{P}\xi_i(t)\|.$$

As $\|\mathscr{P}(n, m)\|=1$, the first term in the right-hand side tends to 0 as $n, m \to \infty$ on account of lemma 1. The second term tends to 0 by virtue of the above condition.

If $\varphi \in H$, $\mathscr{P}\varphi$ and $\mathscr{P}(n, m)\varphi$ are characterized by

$$\mathscr{P}\varphi \in H[C(S_t)],\ \mathscr{P}(n, m)\varphi \in H[C(n, m, S_t)]$$

and

$$\varphi - \mathscr{P}\varphi \perp H[C(S_t)],\ \varphi - \mathscr{P}(n, m)\varphi \perp H[C(n, m, S_t)]$$

or equivalently

$$E\{\varphi - \mathscr{P}\varphi\}\zeta_j(s) = 0,\ E\{\varphi - \mathscr{P}(n, m)\varphi\}\zeta_j(n, m, s) = 0$$

for all $s \in S_t$, $j = 1, \ldots, M$.

LEMMA 4: If $\varphi \in H$, $s \in S_t$, $j = 1, \ldots, M$, then as $n, m \to \infty$,

i) $E\{\mathscr{P}(n, m)\varphi - \mathscr{P}\varphi\}\zeta_j(s) \to 0$ and

ii) $E\{\mathscr{P}(n, m)\varphi - \mathscr{P}\varphi\}\zeta_j(n, m, s) \to 0$.

PROOF: i) Utilizing the above formulae, it is seen that

$$E\{\mathscr{P}(n, m)\varphi - \mathscr{P}\varphi\}\zeta_j(s) = E\{\varphi - \mathscr{P}(n, m)\varphi\}\{\zeta_j(n, m, s) - \zeta_j(s)\}.$$

Because of $\|\varphi - \mathscr{P}(n, m)\varphi\| \leqq 2\|\varphi\|$ it follows from the inequality of Schwarz and from lemma 2 that

$$|E\{\mathscr{P}(n, m)\varphi - \mathscr{P}\varphi\}\zeta_j(s)| \leqq \|\varphi - \mathscr{P}(n, m)\varphi\| \cdot \|\zeta_j(n, m, s) - \zeta_j(s)\| \to 0$$

as $n, m \to \infty$, even uniformly in $s \in S_t$.

ii) Similarly,

$$|E\{\mathscr{P}(n, m)\varphi - \mathscr{P}\varphi\}\zeta_j(n, m, s)| = |E\{\varphi - \mathscr{P}\varphi\}\{\zeta_j(n, m, s) - \zeta_j(s)\}| \leqq$$

$$\leqq 2\|\varphi\| \cdot \|\zeta_j(n, m, s) - \zeta_j(s)\| \to 0 \text{ as } n, m \to \infty, \text{ uniformly in } s \in S_t.$$

According to lemma 3, $\hat{\xi}(t|S_t)$ is stable if

$$E\{\mathscr{P}(n, m)\varphi - \mathscr{P}\varphi\}^2 \to 0 \text{ as } n, m \to \infty \text{ for all } \varphi \in H.$$

Hence $\hat{\xi}(t|S_t)$ is stable if both

$$E\{\mathscr{P}(n, m)\varphi - \mathscr{P}\varphi\}\mathscr{P}\varphi \to 0 \text{ and}$$

$$E\{\mathscr{P}(n, m)\varphi - \mathscr{P}\varphi\}\mathscr{P}(n, m)\varphi \to 0 \text{ as } n, m \to \infty. \tag{8}$$

LEMMA 5: If $\varphi \in H$, $E\{\mathscr{P}(n, m)\varphi - \mathscr{P}\varphi\}\mathscr{P}\varphi \to 0$ as $n, m \to \infty$.

PROOF: Since $\mathscr{P}\varphi \in H[C(S_t)]$, $\mathscr{P}\varphi$ is the limit in q.m. of a sequence, whose members are finite linear combinations of elements of $C(S_t)$. Hence, given $\varepsilon > 0$, there is a decomposition

$$\mathscr{P}\varphi = \zeta + \psi$$

such that

$$\zeta = \sum_{j=1}^{M} \sum_{k=1}^{K} a_{jk}\, \zeta_j(s_k) \text{ and } \|\psi\| < \varepsilon,$$

where the coefficients a_{jk} are real numbers and $s_k \in S_t$, $k=1, \ldots, K$. So

$$|E\{\mathscr{P}(n,m)\varphi-\mathscr{P}\varphi\}\mathscr{P}\varphi| \leqq |E\{\mathscr{P}(n,m)\varphi-\mathscr{P}\varphi\}\zeta| + |E\{\mathscr{P}(n,m)\varphi-\mathscr{P}\varphi\}\psi| \leqq$$

$$\leqq \sum_{j=1}^{M}\sum_{k=1}^{K} |a_{jk}|\cdot|E\{\mathscr{P}(n,m)\varphi-\mathscr{P}\varphi\}\zeta_j(s_k)| + |E\{\mathscr{P}(n,m)\varphi-\mathscr{P}\varphi\}\psi|.$$

The first term in the right-hand side tends to 0 as $n, m \to \infty$ by virtue of statement i) of lemma 4. And on account of the inequality of Schwarz,

$$|E\{\mathscr{P}(n,m)\varphi-\mathscr{P}\varphi\}\psi| \leqq \|\mathscr{P}(n,m)\varphi-\mathscr{P}\varphi\|\cdot\|\psi\| \leqq 2\|\varphi\|\varepsilon.$$

As a corollary of this lemma we obtain

LEMMA 6: In order that $\hat{\xi}(t|S_t)$ be stable it is sufficient that condition (8) is satisfied.

Concerning (8), an approach similar to that in lemma 5 breaks down, in spite of ii) in lemma 4. The result in section 4 shows that (8) is not true in general.

3. THE STABILITY OF $\hat{\xi}(t|S_t)$ IF S_t IS A FINITE SET

Let $t \in [0, T]$ be a fixed value and let $S_t = \{s_k, k=0, 1, \ldots, K\}$ be such that $0=s_0<s_1<\ldots<s_K \leqq T$. It is for sake of convenience that we assume $0 \in S_t$. This may be done without loss of generality since the observations are equal to 0 at $t=0$, see also (6).

Now the classes $C(S_t)$ and $C(n, m, S_t)$ are finite. Hence the spaces

$$H[C(S_t)] \text{ and } H[C(n, m, S_t)]$$

are finite dimensional, Euclidean. They may also be generated by the elements of the classes of differences

$$D(S_t) = \{\zeta_{jk} = \zeta_j(s_k) - \zeta_j(s_{k-1}),\ j=1, \ldots, M,\ k=1, \ldots, K\}$$

and

$$D(n, m, S_t) = \{\zeta_{jk}(n, m) = \zeta_j(n, m, s_k) - \zeta_j(n, m, s_{k-1}), j=1, \ldots, M, k=1, \ldots, K\}$$

respectively, since $\zeta(0)=\zeta(n, m, 0)=0$. As we also introduce

$$\eta_{jk}(n) = \eta_j(n, s_k) - \eta_j(n, s_{k-1})$$

and

$$\beta_{jk}(m) = \beta_j(m, s_k) - \beta_j(m, s_{k-1})$$

then

$$\zeta_{jk}(n, m) = \eta_{jk}(n) + \beta_{jk}(m),$$

$j=1, \ldots, M,\ k=1, \ldots, K,\ n, m=1, 2, \ldots.$

According to lemma 4, statement ii),

$$E\{\mathscr{P}(n, m)\varphi - \mathscr{P}\varphi\}\zeta_{jk}(n, m) \to 0 \text{ as } n, m \to \infty. \tag{9}$$

In this case, where S_t is a finite set, it can be shown that $\hat{\xi}(t|S_t)$ is stable. Owing to lemma 6 we only need to show that condition (8),

$$E\{\mathscr{P}(n, m)\varphi - \mathscr{P}\varphi\}\mathscr{P}(n, m)\varphi \to 0 \text{ as } n, m \to \infty,$$

is fulfilled. As $\mathscr{P}(n, m)\varphi$ is an element of the Euclidean space

$$H[C(n, m, S_t)] = H[D(n, m, S_t)]$$

it may be decomposed as

$$\mathscr{P}(n, m)\varphi = \sum_{j=1}^{M} \sum_{k=1}^{K} c_{jk}(n, m)\, \zeta_{jk}(n, m) \tag{10}$$

where the coefficients $c_{jk}(n, m)$ are real numbers. They are not necessarily unique, since the elements $\zeta_{jk}(n, m)$ might not be linearly independent (if n and m are small). We shall first establish the following lemma.

Lemma 7: Given $\varphi \in H$, there is a number r, such that the coefficients $c_{jk}(n, m)$ in (10) are bounded, uniformly in n and in $m > r$.

Proof: We may write

$$\mathscr{P}(n, m)\varphi = \sum_{j=1}^{M} \sum_{k=1}^{K} c_{jk}(n, m)\, \eta_{jk}(n) + \sum_{j=1}^{M} \sum_{k=1}^{K} c_{jk}(n, m)\, \beta_{jk}(m).$$

The two terms in the right-hand side are orthogonal because of (7). Hence

$$\|\mathscr{P}(n, m)\, \varphi\|^2 = \|\sum_{j=1}^{M} \sum_{k=1}^{K} c_{jk}(n, m)\, \eta_{jk}(n)\|^2 + \|\sum_{j=1}^{M} \sum_{k=1}^{K} c_{jk}(n, m)\, \beta_{jk}(m)\|^2$$

and so

$$\|\varphi\|^2 \geqq \|\mathscr{P}(n, m)\varphi\|^2 \geqq \|\sum_{j=1}^{M} \sum_{k=1}^{K} c_{jk}(n, m)\beta_{jk}(m)\|^2.$$

Let us introduce the MK-vector $c(n, m)$,

$$c^{\mathsf{T}}(n, m) = (c_{11}(n, m) \dots c_{M1}(n, m), \dots, c_{1K}(n, m) \dots c_{MK}(n, m)),$$

and the covariance $MK \times MK$-matrix $C(m)$,

$$C(m) = \begin{pmatrix} C_{11}(m) & \dots & C_{1K}(m) \\ \vdots & & \vdots \\ C_{K1}(m) & \dots & C_{KK}(m) \end{pmatrix}$$

whose $M \times M$-submatrices $C_{kh}(m)$, $k, h = 1, \dots, K$, are defined as

$$C_{kh}(m) = E\begin{pmatrix} \beta_{1k}(m) \\ \vdots \\ \beta_{Mk}(m) \end{pmatrix}(\beta_{1h}(m) \dots \beta_{Mh}(m)) =$$

$$= E(\beta(m, s_k) - \beta(m, s_{k-1}))\,(\beta(m, s_h) - \beta(m, s_{h-1}))^{\mathsf{T}}.$$

Then the above inequality reeds

$$(11)\qquad \|\varphi\|^2 \geqq c^{\mathrm{T}}(n, m)C(m)c(n, m).$$

Owing to the properties v) and iii) in theorem 1 it follows, as $m \to \infty$, that the elements of $C_{kh}(m)$ tend to 0 if $k \neq h$, and to the corresponding elements of the $M \times M$-matrix $\int_{s_{k-1}}^{s_k} B(s)ds$ if $k=h$. Hence, if $m \to \infty$, the elements of $C(m)$ tend to the corresponding elements of the covariance $MK \times MK$-matrix C, defined as

$$C = \begin{pmatrix} \int_0^{s_1} B(s)\,ds & & (0) \\ & \ddots & \\ (0) & & \int_{s_{K-1}}^{s_K} B(s)\,ds \end{pmatrix}$$

By virtue of (1), $\int_{s_{k-1}}^{s_k} B(s)ds \geqq e(s_k - s_{k-1})I_M$, and hence

$$C \geqq edI_{MK} > 0,\ d = \min_{k=1,..,K} (s_k - s_{k-1}).$$

Since the elements of $C(m)$ converge to the corresponding elements of C as $m \to \infty$, it is easily seen that there is a number r, such that

$$C(m) \geqq \tfrac{1}{2} edI_{MK} \text{ as } m > r,$$

and so

$$c^{\mathrm{T}}(n, m)C(m)c(n, m) \geqq \tfrac{1}{2} edc^{\mathrm{T}}(n, m)c(n, m) \text{ as } m > r.$$

Hence, on account of (11),

$$\sum_{j=1}^{M} \sum_{k=1}^{K} c^2_{jk}(n, m) \leqq \frac{2}{ed} \|\varphi\| \text{ as } m > r, \text{ independent of } n,$$

finishing the proof.

THEOREM 2: If the number of observations is finite, i.e. if S_t is a finite set, then $\hat{\xi}(t|S_t)$ is stable in the sense of definition 2, independent of the position of t and S_t in $[0, T]$.

PROOF: Given $\varphi \in H$, let c be the bound of the coefficients $c_{jk}(n, m)$, $n = 1, 2, \ldots, m > r$, established in the above lemma. Then, on account of (10) and (9), and as $m > r$,

$$|E\{\mathscr{P}(n, m)\varphi - \mathscr{P}\varphi\}\mathscr{P}(n, m)\varphi| = |\sum_{j=1}^{M} \sum_{k=1}^{K} c_{jk}(n,m)E\{\mathscr{P}(n,m)\varphi - \mathscr{P}\varphi\}\zeta_{jk}(n,m)| \leqq$$

$$\leqq c \cdot \sum_{j=1}^{M} \sum_{k=1}^{K} |E\{\mathscr{P}(n, m)\varphi - \mathscr{P}\varphi\}\zeta_{jk}(n, m)| \to 0 \text{ as } n, m \to \infty.$$

Owing to lemma 6 the proof is finished.

4. THE BEHAVIOUR OF $\hat{\xi}(n, m, t|S_t)$ IF S_t IS AN INFINITE SET

Let us now consider the situation that the number of observations is infinite, i.e. that S_t is an infinite set. We shall show that in this case $\hat{\xi}(t|S_t)$ is not necessarily stable in the sense of definition 2.

Let $n, m, t \in [0, T]$ and $S_t \subset [0, T]$ be fixed. In accordance with i) in theorem 1, it may and will be assumed that for all $s \in [0, T]$, $\alpha(n, s)$ and $\beta(m, s)$ are defined on one and the same finite set of random variables, say on

$$\alpha_i(n, s_u),\ i=1, \ldots, N,\ u=1, \ldots, U,$$

and

$$\beta_j(m, s'_v),\ j=1, \ldots, M,\ v=1, \ldots, V.$$

Then the components $\xi_i(n, t)$ of $\xi(n, t)$ are linear combinations of the elements of the class

$$C_1=\{\alpha_i(n, s_u),\ \gamma_i,\ i=1, \ldots, N,\ u=1, \ldots, U\}.$$

Hence, if $H[C_1]$ is the Euclidean subspace of H, generated by the elements of C_1, then

$$\xi_i(n, t) \in H[C_1],\ i=1, \ldots, N.$$

And the components $\zeta_j(n, m, s)$ of $\zeta(n, m, s)$, $j=1, \ldots, M$, $s \in S_t$, are linear combinations of the elements of the class

$$C_2=\{\alpha_i(n, s_u), \gamma_i, i=1,\ldots,N, u=1, \ldots, U; \beta_j(m,s'_v), j=1,\ldots,M, v=1,\ldots, V\}.$$

It follows that $C_1 \subset C_2$ and hence

$$H[C_1] \subset H[C_2]$$

where $H[C_2]$ is the Euclidean subspace of H, generated by the elements of C_2.

As S_t is an infinite set, there are given infinitely many elements $\zeta_j(n, m, s)$, $s \in S_t$, $j=1, \ldots, M$, each of which being a linear combination of the elements of the finite class C_2. Then reversed, in non-trivial cases, the elements of C_2 may be written as linear combinations of some of the elements $\zeta_j(n, m, s)$, $s \in S_t$, $j=1, \ldots, M$. This entails that now

$$H[C(n, m, S_t)]=H[C_2]$$

and

$$\xi_i(n, t) \in H[C_1] \subset H[C_2].$$

As $\hat{\xi}_i(n, m, t|S_t)$ is the orthogonal projection of $\xi_i(n, t)$ onto $H[C(n, m, S_t)]$, it follows that in this case $\hat{\xi}_i(n, m, t|S_t)$ is the orthogonal projection of $\xi_i(n, t)$ onto the space $H[C_2]$ of which $\xi_i(n, t)$ itself is an element. Hence

$$\hat{\xi}_i(n, m, t|S_t)=\xi_i(n, t).$$

This result may be true at each $n, m=1, 2, \ldots$. Then it follows by virtue of lemma 1 that

$$\hat{\xi}_i(n, m, t|S_t)=\xi_i(n, t) \rightarrow \xi_i(t) \text{ as } n \text{ (and } m) \rightarrow \infty.$$

As in non-trivial cases $\hat{\xi}(t|S_t) \neq \xi(t)$ (see the Kalman-Bucy filter), we have shown that it may occur that

$$\hat{\xi}_i(n, m, t|S_t) \not\to \hat{\xi}_i(t|S_t) \text{ as } n, m \to \infty,$$

i.e. that $\hat{\xi}(t|S_t)$ is not stable if S_t is an infinite set.

5. Conclusion

Given the infinite set $S_t \subset [0, T]$, let

$$\{S_{tk},\ k=1, 2, \dots\}$$

be a sequence of finite subsets of S_t, increasing to (a set dense in) S_t as $k \to \infty$. Since $\zeta_j(s)$ and $\zeta_j(n, m, s)$, $j=1, \dots, M$, are continuous in q.m. as functions of $s \in [0, T]$, it follows from well-known properties of sequences of increasing orthogonal projectors that at fixed n, m and t,

$$\hat{\xi}_i(t|S_{tk}) \to \hat{\xi}_i(t|S_t) \text{ in q.m.}$$

and

$$\hat{\xi}_i(n, m, t|S_{tk}) \to \hat{\xi}_i(n, m, t|S_t) \text{ in q.m.}$$

as $k \to \infty$, $i=1, \dots, N$.

Assuming that n and m pass to infinity through the sequences

$$\{n_1, n_2, \dots\} \text{ and } \{m_1, m_2, \dots\}$$

respectively, we may gather the results of this paper in the following diagram, where the arrow stands for "converges in q.m. to".

$$\begin{array}{cccc}
\hat{\xi}(n_1, m_1, t|S_{t1}) & \hat{\xi}(n_2, m_2, t|S_{t1}) & \dots\dots \to & \hat{\xi}(t|S_{t1}) \\
\hat{\xi}(n_1, m_1, t|S_{t2}) & \hat{\xi}(n_2, m_2, t|S_{t2}) & \dots\dots \to & \hat{\xi}(t|S_{t2}) \\
\dots\dots\dots & \dots\dots\dots & \dots\dots & \dots\dots \\
\downarrow & \downarrow & & \downarrow \\
\hat{\xi}(n_1, m_1, t|S_t) & \hat{\xi}(n_2, m_2, t|S_t) & \dots\dots \not\to & \hat{\xi}(t|S_t)
\end{array}$$

The convergence along the columns is shown above. The behaviour along the last row is explained in the previous section. The convergence along the other rows is established in section 3.

The convergence along the first rows, i.e. the stability in the sense of definition 2 of $\hat{\xi}(t|S_{tk})$ where S_{tk} is a finite set, could be established owing to condition (1), imposed on the Wiener-Lévy process $\beta(t)$ in the observations. This is the same condition on which depends the success of the Kalman-Bucy filter.

The last row shows that $\hat{\xi}(t|S_t)$ is in general not stable in the sense of definition 2 if S_t is an infinite set. However, it may be approximated by the stable estimates $\hat{\xi}(t|S_{tk})$. Hence (setting $S_t = [0, t]$), it follows that the meaning of the Kalman-Bucy estimate is more or less controversial.

Department of Mathematics
University of Technology, Delft
The Netherlands

REFERENCES

1. Ruymgaart, P. A., The Kalman-Bucy filter and its behaviour with respect to smooth perturbations of the involved Wiener-Lévy processes, Thesis (1971).